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2005 J. Phys. A: Math. Gen. 38 6405

(http://iopscience.iop.org/0305-4470/38/28/013)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **38** (2005) 6405–6418 doi:10.1088/0305-4470/38/28/013

Toric Calabi–Yau supermanifolds and mirror symmetry

A Belhaj[1], **L B Drissi**[2], **J Rasmussen**[3], **E H Saidi**[2] **and A Sebbar**[1]

[1] Department of Mathematics and Statistics, Ottawa University, 585 King Edward Avenue, Ottawa, ON K1N 6N5, Canada
[2] National Grouping of High Energy Physics, GNPHE and Lab/UFR HEP, Department of Physics, Faculty of Sciences, Rabat, Morocco
[3] Centre de Recherches Mathématiques, Université de Montréal, C.P. 6128, succursale centre-ville, Montréal, QC H3C 3J7, Canada

E-mail: abelh633@mathstat.uottawa.ca, rasmusse@crm.umontreal.ca, esaidi@ictp.trieste.it and sebbar@mathstat.uottawa.ca

Received 13 January 2005
Published 29 June 2005
Online at stacks.iop.org/JPhysA/38/6405

Abstract

We study mirror symmetry of supermanifolds constructed as fermionic extensions of compact toric varieties. We mainly discuss the case where the linear sigma A-model contains as many fermionic fields as there are $U(1)$ factors in the gauge group. In the mirror super-Landau–Ginzburg B-model, focus is on the bosonic structure obtained after integrating out all the fermions. Our key observation is that there is a relation between the super-Calabi–Yau conditions of the A-model and quasi-homogeneity of the B-model, and that the degree of the associated superpotential in the B-model is given in terms of the determinant of the fermion charge matrix of the A-model.

PACS numbers: 11.25.−w, 02.40.Tt, 11.30.Pb

1. Introduction

Mirror symmetry underlies one of the most important and interesting examples of string dualities and provides a symmetry between Calabi–Yau (CY) manifolds interpreted in terms of closed topological string theories. More generally, the so-called A- and B-models are related by mirror symmetry, as discussed below. It has been realized, though, that rigid CY manifolds can have mirror manifolds which are not themselves CY geometries. An intriguing remedy is the introduction of CY *supermanifolds* in these considerations [1, 2]. It has thus been suggested that mirror symmetry is between supermanifolds and manifolds alike, and not just between bosonic manifolds.

It has been found recently that there is a correspondence between the moduli space of holomorphic Chern–Simons theory on the CY supermanifold $\mathbf{CP}^{3|4}$ and, self-dual, four-dimensional $N = 4$ Yang–Mills theory [3, 4]. This may also be related to the B-model of open topological string theory having $\mathbf{CP}^{3|4}$ as target space. Partly based on this work, CY supermanifolds have subsequently attracted a great deal of attention [5–11]. It has been found, for instance, that an A-model defined on the CY supermanifold $\mathbf{CP}^{3|4}$ is a mirror of a B-model on a quadric hypersurface in $\mathbf{CP}^{3|3} \times \mathbf{CP}^{3|3}$, provided the Kähler parameter of $\mathbf{CP}^{3|4}$ approaches infinity [5, 6]. Following this observation, a possible generalization of the A-model has been considered in which fermionic coordinates with *different* weights are introduced without changing the bosonic manifold $\mathbf{CP}^3$ [8, 9].

The aim of the present work is to study mirror symmetry based on a broad class of supermanifolds whose bosonic parts correspond to compact toric varieties. Important examples of such bosonic manifolds are (weighted) projective spaces and products thereof.

Our analysis is based on the following scenario. The bosonic part of the A-model is constructed as a $U(1)^{\otimes n}$ linear sigma model whose target space is a toric variety. Adding a set of f fermionic fields with charges given by an $n \times f$ matrix to the sigma model, corresponds to extending the toric variety to a supermanifold with f Grassmannian coordinates. By extending the T-duality prescription in [6] on fermionic fields to cover the *product* gauge group $U(1)^{\otimes n}$, we can obtain the path integral description of the mirror super-Landau–Ginzburg (super-LG) B-model. It initially involves n delta functions which may be integrated out to extract information on the associated (super-)geometry. We shall focus on the bosonic structure obtained after integrating out all the fermionic fields in the B-model. Different patches may result depending on which bosonic fields are integrated in the elimination of the delta functions. We pay particular attention to the 'quadratic' case where $f = n$, and consider generic values of the Kähler parameters, that is, we do not restrict ourselves to simplifying limits.

Our key observation in this set-up is that the super-CY conditions of the A-model geometry are related to quasi-homogeneity of the bosonic toric data of the B-model. Furthermore, the degree of the associated quasi-homogeneous superpotential in the B-model is given in terms of the determinant of the matrix of fermion charges in the A-model. Details on this correspondence and the dependence on the determinant will be provided in the main text.

It is emphasized that the explicit mirrors of supermanifolds discussed in [6, 7, 9], for example, all have fewer gauge-group factors than fermionic fields in the A-model, i.e., $n < f$. As already mentioned, the detailed part of our analysis pertains to the quadratic case where $n = f$. We are thus not concerned here with reproducing the existing results, while we intend to discuss elsewhere the cases where $n \neq f$.

After a brief summary of T-duality for fermionic coordinates or fields in section 2, we discuss mirror symmetry of supermanifolds in section 3. Our main result involving the super-CY conditions and the determinant of fermion charges is derived for products of weighted super-projective spaces. We emphasize the situation for complex three-dimensional projective spaces due to their relevance in string theory. We also relate our results based on $\mathbf{CP}^1 \times \mathbf{CP}^{p-1}$ to superpotentials discussed in [12]. Section 4 concerns the extension to general toric varieties, and we find that our key observation still holds. The family $\{\tilde{F}_m\}$ of three-dimensional toric varieties generalizing the projective spaces are used as an illustration. A conclusion is presented in section 5.

2. *T*-duality of fermionic fields

In this section we review T-duality for fermionic coordinates (fields), and we do this by first recalling the bosonic case [13].

To this end, we consider a linear sigma model described in terms of the chiral fields Φ_i, $i = 1, \ldots, p$, with charges q_i under a $U(1)$ gauge symmetry [14]. The requirement of conformal invariance of this system is equivalent to the CY condition of the target space,

$$\sum_i q_i = 0. \tag{1}$$

The vanishing condition for the potential energy density for the scalar fields reads

$$\sum_i q_i |\Phi_i|^2 = r, \tag{2}$$

where r is a Fayet–Iliopoulos (FI) coupling constant which, combined with the $U(1)$ θ-angle, defines the complexified Kähler parameter $t = r + \mathrm{i}\theta$. Note in passing that equation (2) corresponds to a local CY manifold.

Following [13, 15, 16], the mirror model is obtained by introducing a set of fields $\{Y_i\}$ dual to the set $\{\Phi_i\}$, such that

$$\Re(Y_i) = |\Phi_i|^2, \tag{3}$$

where $\Re(Y_i)$ denotes the real part of Y_i. The mirror version of (2) is

$$\sum_i q_i Y_i = t \tag{4}$$

and the corresponding superpotential in the associated LG model reads

$$W = \sum_i \mathrm{e}^{-Y_i}. \tag{5}$$

Using the following field redefinitions:

$$\hat{y}_i = \mathrm{e}^{-Y_i}, \tag{6}$$

the superpotential becomes

$$W = \sum_i \hat{y}_i, \tag{7}$$

subject to

$$\prod_i \hat{y}_i^{q_i} = \mathrm{e}^{-t}. \tag{8}$$

With multiple toric actions, $U(1)^{\otimes n}$, this extends readily to

$$\prod_i \hat{y}_i^{q_i^a} = \mathrm{e}^{-t^a}, \qquad a = 1, \ldots, n. \tag{9}$$

Here t^a is the complexified Kähler parameter associated with the ath $U(1)$ factor, while q^a is the charge vector with respect to the same $U(1)$ factor.

It has been shown recently that a similar analysis can be carried out for *fermionic* fields as well, though with a different rule for 'dualizing' the fields [6]. For a system with fermionic fields $\{\Psi_\alpha\}$ with charges Q_α, and bosonic fields $\{\Phi_i\}$ as above, the extended D-term constraint of equation (2) reads

$$\sum_i q_i |\Phi_i|^2 + \sum_\alpha Q_\alpha |\Psi_\alpha|^2 = \Re(t). \tag{10}$$

The condition for the associated super-variety to be a CY supermanifold is given by

$$\sum_i q_i = \sum_\alpha Q_\alpha. \tag{11}$$

Under T-duality, the bosonic superfield Φ_i of the linear sigma model is still replaced by a dual superfield Y_i. The fermionic superfield Ψ_α, on the other hand, is dualized by the triplet $(X_\alpha, \eta_\alpha, \chi_\alpha)$ [6], where the bosonic field X_α satisfies

$$\Re(X_\alpha) = -|\Psi_\alpha|^2. \tag{12}$$

The accompanying pair of fields, η_α and χ_α, are fermionic and are required to preserve the superdimension and hence the total central charge under the symmetry. The corresponding mirror super-LG model is given by the path integral

$$\mathcal{Z} = \int \left(\prod_i \mathrm{d}Y_i\right)\left(\prod_\alpha \mathrm{d}X_\alpha\,\mathrm{d}\eta_\alpha\,\mathrm{d}\chi_\alpha\right)\delta\left(\sum_i q_i Y_i - \sum_\alpha Q_\alpha X_\alpha - t\right) \\ \times \exp\left(\sum_i \mathrm{e}^{-Y_i} + \sum_\alpha \mathrm{e}^{-X_\alpha}(1+\eta_\alpha\chi_\alpha)\right). \tag{13}$$

The objective in the following is to extend this analysis to a linear sigma A-model with *product* gauge group $U(1)^{\otimes n}$ and f fermionic fields, and study the resulting mirror B-model as defined by a generalization of (13). One may in this case supplement the field redefinitions in (6) with

$$\hat{x}_\alpha = \mathrm{e}^{-X_\alpha}, \qquad \alpha = 1,\ldots,f. \tag{14}$$

The associated conditions (9) on the bosonic part of the B-model superpotential then read

$$\prod_i \hat{y}_i^{q_i^a} = \mathrm{e}^{-t^a}\prod_\alpha \hat{x}^{Q_\alpha^a}, \qquad a = 1,\ldots,n. \tag{15}$$

Focus will be on the toric data of the bosonic structure obtained after eliminating the n delta functions and integrating out the $2f$ fermionic fields in the B-model path integral. We shall find that this bosonic structure is described in terms of the set of fermion charges in the A-model. In particular, the super-CY conditions of the A-model extending (11) turn out to be related to quasi-homogeneity of the bosonic structure of the B-model.

3. Mirrors of super-projective spaces

We recall that a general complex p-dimensional toric variety can be expressed in the following form:

$$\mathbf{V}^p = \frac{\mathbb{C}^{p+n}\backslash S}{\mathbb{C}^{*n}}, \tag{16}$$

where the n $\mathbb{C}^*$ actions are given by

$$\mathbb{C}^{*n} : z_i \to \lambda^{q_i^a} z_i, \qquad i = 1,\ldots,p+n, \quad a = 1,\ldots,n. \tag{17}$$

In these expressions, the exponents q_i^a are referred to as charges and are assumed to be integers. For each fixed a, they define a Mori vector in toric geometry. These vectors thus generalize the weight vector w of the weighted projective space $\mathbf{WCP}^p_{(w_1,\ldots,w_{p+1})}$. The subtracted part S is a subset of $\mathbb{C}^{p+n}$ chosen by triangulation. The variety $\mathbf{V}^p$ can be represented by a toric diagram $\Delta(\mathbf{V}^p)$ spanned by $k = p+n$ vertices v_i in a $\mathbb{Z}^p$ lattice satisfying

$$\sum_{i=1}^{p+n} q_i^a v_i = 0, \qquad a = 1,\ldots,n. \tag{18}$$

It may be realized in terms of an $N = 2$ linear sigma model, where one considers a two-dimensional supersymmetric $N = 2$ gauge system with $U(1)^{\otimes n}$ gauge group and $p+n$ chiral fields Φ_i with a charge matrix whose entries are q_i^a [14]. In this way, and up to $U(1)^{\otimes n}$ gauge transformations, the Kähler manifold $\mathbf{V}^p$ corresponds to the minimum of the D-term potential $(D^{a=0})$. That is,

$$\sum_{i=1}^{p+n} q_i^a |\Phi_i|^2 = r^a, \tag{19}$$

where the r^a are FI coupling parameters.

Here we consider the complex p-dimensional toric variety defined by the following trivial fibration:

$$\mathbf{WCP}^{p_1-1}_{(w_1^1,\ldots,w_{p_1}^1)} \times \mathbf{WCP}^{p_2-1}_{(w_1^2,\ldots,w_{p_2}^2)} \times \cdots \times \mathbf{WCP}^{p_n-1}_{(w_1^n,\ldots,w_{p_n}^n)}, \tag{20}$$

where $p = \sum_{a=1}^n (p_a - 1)$. It admits a $U(1)^{\otimes n}$ sigma-model description in terms of the $p+n$ bosonic fields

$$\left\{\Phi_1^1, \ldots, \Phi_{p_1}^1; \Phi_1^2, \ldots, \Phi_{p_2}^2; \ldots; \Phi_1^n, \ldots, \Phi_{p_n}^n\right\}, \tag{21}$$

with charge vectors

$$\begin{aligned} q^1 &= \left(w_1^1, \ldots, w_{p_1}^1; 0, \ldots, 0; \ldots; 0, \ldots, 0\right), \\ q^2 &= \left(0, \ldots, 0; w_1^2, \ldots, w_{p_2}^2; 0, \ldots, 0; \ldots; 0, \ldots, 0\right), \\ &\vdots \\ q^n &= \left(0, \ldots, 0; \ldots; 0, \ldots, 0; w_1^n, \ldots, w_{p_n}^n\right). \end{aligned} \tag{22}$$

A toric variety like (20) is compact if all the charges (22) are positive (or negative) integers. Its associated sigma model is a solution of the D-term constraints

$$\sum_{i=1}^{p_a} w_i^a \left|\Phi_i^a\right|^2 = \Re(t^a), \qquad a = 1, \ldots, n, \tag{23}$$

where t^a is the complexified Kähler parameter associated with the ath factor in (20). All weights w_i^a are assumed non-vanishing. By convention for weighted projective spaces, the greatest common divisor of the weights w_i^a for a given a is 1.

The objective now is to consider a fermionic extension of the manifold (20), thus turning it into a (weighted) super-projective space and discuss its mirror companion. Our approach may be seen as an illustration and an extension of the previous section by taking into account the product structure of (20) with its enlarged symmetry.

Adding f Grassmann coordinates to (20) corresponds to supplementing the bosonic sigma model, described by (21), by f fermionic fields,

$$\{\Psi_\alpha, \alpha = 1, \ldots, f\}, \tag{24}$$

with charges Q_α^a. The full spectrum of $U(1)^{\otimes n}$ charge vectors thus becomes

$$\begin{aligned} q'^1 &= \left(q^1 \,\middle|\, Q_1^1, \ldots, Q_f^1\right), \\ q'^2 &= \left(q^2 \,\middle|\, Q_1^2, \ldots, Q_f^2\right), \\ &\vdots \\ q'^n &= \left(q^n \,\middle|\, Q_1^n, \ldots, Q_f^n\right), \end{aligned} \tag{25}$$

while the extended D-term constraints of this A-model (cf (23)) read

$$\sum_{i=1}^{p_a} w_i^a \left|\Phi_i^a\right|^2 + \sum_{\alpha=1}^{f} Q_\alpha^a |\Psi_\alpha|^2 = \Re(t^a), \qquad a = 1, \ldots, n. \tag{26}$$

There is an abundance of possible fermionic extensions following this prescription. It may be limited, though, by imposing the super-CY conditions (11):

$$0 = \sum_i q_i^a - \sum_\alpha Q_\alpha^a = \sum_{i=1}^{p_a} w_i^a - \sum_{\alpha=1}^{f} Q_\alpha^a, \qquad a = 1, \ldots, n. \tag{27}$$

We shall initially refrain from doing this since one of our key observations will be that these conditions are related to quasi-homogeneity of a bosonic structure of the B-model geometry. This new correspondence between two mirror supermanifolds will be addressed below.

Before proceeding, we recall that the charge vectors (22) and D-term constraints (23) of the bosonic A-model correspond to a sigma-model realization of the toric variety (20). This extends readily to general toric varieties as defined by (16), and we shall have more to say about this in section 4. Here we wish to point out that in a similar fashion the expressions (25) and (26) may be seen as corresponding to an $N = 2$ sigma-model realization of a *fermionic* extension of a toric variety. A *super-variety*

$$\mathbf{V}^{p|f} = \frac{\mathbb{C}^{p+n|f} \backslash S}{\mathbb{C}^{*n}} \tag{28}$$

is thereby represented by a toric super-diagram spanned by $(p+n)$ vertices v_i and f vertices v_α in a superlattice $\mathbb{Z}^{p|f}$, and constrained as

$$\sum_{i=1}^{p+n} q_i^a v_i - \sum_{\alpha=1}^{f} Q_\alpha^a v_\alpha = 0, \qquad a = 1, \ldots, n. \tag{29}$$

Here we have used the notation $\mathbb{C}^{p+n|f}$ to indicate a fermionic extension of $\mathbb{C}^{p+n}$. We suggest to refer to this fermionic extension of toric geometry as toric super-geometry. It is seen that CY supermanifolds are defined naturally in toric super-geometry.

According to the T-duality outlined in the previous section, the mirror B-model is now obtained by replacing the field Φ_i^a by a superfield Y_i^a, while the fermionic field Ψ_α is dualized by the triplet $(X_\alpha, \eta_\alpha, \chi_\alpha)$. Applying the mirror symmetry transformation to the A-model above thus results in a B-model in terms of a super-LG mirror model given by the path integral

$$\begin{aligned} \mathcal{Z} = \int \left(\prod_{i=1}^{p_1} \mathrm{d}Y_i^1\right) \cdots \left(\prod_{i=1}^{p_n} \mathrm{d}Y_i^n\right) \left(\prod_{\alpha=1}^{f} \mathrm{d}X_\alpha\, \mathrm{d}\eta_\alpha\, \mathrm{d}\chi_\alpha\right) \delta\left(\sum_{i=1}^{p_1} w_i^1 Y_i^1 - \sum_{\alpha=1}^{f} Q_\alpha^1 X_\alpha - t^1\right) \\ \times \cdots \times \delta\left(\sum_{i=1}^{p_n} w_i^n Y_i^n - \sum_{\alpha=1}^{f} Q_\alpha^n X_\alpha - t^n\right) \\ \times \exp\left(\sum_{a=1}^{n} \sum_{i=1}^{p_a} \mathrm{e}^{-Y_i^a} + \sum_{\alpha=1}^{f} \mathrm{e}^{-X_\alpha}(1 + \eta_\alpha \chi_\alpha)\right). \end{aligned} \tag{30}$$

To extract information on the B-model (super-)geometry, one would naturally wish to integrate out the n delta functions. In this paper, we shall focus on the 'quadratic' case where $n = f$ and subsequently choose to integrate out the bosonic fields X_α, $\alpha = 1, \ldots, f = n$. We intend to address elsewhere the situations where $f \neq n$ or where the elimination of the delta functions may involve integrating out some of the fields Y_i. As already mentioned, we are here interested in the bosonic structure arising after integrating out all the $2f$ fermionic fields.

To illustrate the construction, focus here will be on the situation where $n = 2$, $p_1 = 2$ and $p_2 = 3$, while in section 4 we shall report on the case based on a general toric variety with $f = n$. For now, our chosen A-model scenario thus corresponds to a fermionic extension of the complex three-dimensional variety $\mathbf{WCP}^1_{(w_1^1,w_2^1)} \times \mathbf{WCP}^2_{(w_1^2,w_2^2,w_3^2)}$ which we shall assume is compact. A particular example is provided by $\mathbf{CP}^1 \times \mathbf{CP}^2$ and corresponds to a trivial fibration of $\mathbf{CP}^1$ over the base space $\mathbf{CP}^2$. This manifold is sometimes denoted by $\tilde{F}_0$ and has been used in the construction of real four-dimensional $N = 1$ models obtained from F-theory compactification on elliptic CY fourfolds [17]. We shall have more to say about the infinite family of complex three-dimensional manifolds $\tilde{F}_m$, as it appears as a particular subclass of the general study in section 4.

3.1. Mirrors of fermionic extensions of $\mathbf{WCP}^1_{(w_1^1,w_2^1)} \times \mathbf{WCP}^2_{(w_1^2,w_2^2,w_3^2)}$

We are here considering the case with $f = n = 2$. The integral over the four fermionic fields η_1, η_2, χ_1 and χ_2 appearing in the super-LG B-model (30) produces a simple expression in X_1 and X_2:

$$\int \left(\prod_{\alpha=1}^{2} \mathrm{d}\eta_\alpha\, \mathrm{d}\chi_\alpha\right) \exp(\mathrm{e}^{-X_1}(1+\eta_1\chi_1) + \mathrm{e}^{-X_2}(1+\eta_2\chi_2)) = \mathrm{e}^{-X_1}\,\mathrm{e}^{-X_2} \exp(\mathrm{e}^{-X_1} + \mathrm{e}^{-X_2}). \tag{31}$$

Solving the delta-function constraints amounts to solving the two linear equations

$$\begin{aligned} Q_1^1 X_1 + Q_2^1 X_2 &= w_1^1 Y_1^1 + w_2^1 Y_2^1 - t^1, \\ Q_1^2 X_1 + Q_2^2 X_2 &= w_1^2 Y_1^2 + w_2^2 Y_2^2 + w_3^2 Y_3^2 - t^2. \end{aligned} \tag{32}$$

There is a unique solution for X_1 and X_2, provided the determinant

$$D = Q_1^1 Q_2^2 - Q_2^1 Q_1^2 \tag{33}$$

is non-vanishing. For vanishing determinant, the equations (32) would impose linear relations among the fields $\{Y_i^a\}$. We shall assume that $D \neq 0$, in which case the path integral (30) reduces to

$$\begin{aligned} \mathcal{Z} \propto \int \left(\prod_{i=1}^{2} \mathrm{d}Y_i^1\right)\left(\prod_{i=1}^{3} \mathrm{d}Y_i^2\right) \mathrm{e}^{\frac{Q_1^2-Q_2^2}{D}\sum_{i=1}^{2} w_i^1 Y_i^1 - \frac{Q_1^1-Q_2^1}{D}\sum_{i=1}^{3} w_i^2 Y_i^2} \\ \times \exp\left(\sum_{i=1}^{2} \mathrm{e}^{-Y_i^1} + \sum_{i=1}^{3} \mathrm{e}^{-Y_i^2}\right) \exp\left(\mathrm{e}^{-\frac{Q_2^2}{D}[\sum_{i=1}^{2} w_i^1 Y_i^1 - t^1] + \frac{Q_2^1}{D}[\sum_{i=1}^{3} w_i^2 Y_i^2 - t^2]}\right) \\ \times \exp\left(\mathrm{e}^{\frac{Q_1^2}{D}[\sum_{i=1}^{2} w_i^1 Y_i^1 - t^1] - \frac{Q_1^1}{D}[\sum_{i=1}^{3} w_i^2 Y_i^2 - t^2]}\right). \end{aligned} \tag{34}$$

Our current objective is to extract information on the geometry associated with the path integral (34). This may be achieved naively if one can make field redefinitions turning the path integral into the form

$$\mathcal{Z} \simeq \int \left(\prod_{k}^{\ell} \mathrm{d}\varphi_k\right) \mathrm{e}^{-W(\{\varphi_k\})}. \tag{35}$$

The functional expression W is then referred to as the superpotential, and its vanishing condition, $W = 0$, provides an algebraic equation in terms of $\{\varphi_k\}$. For it to correspond to a conventional LG theory, it should be quasi-homogeneous in the sense that

$$W(\{\lambda^{w_k}\varphi_k\}) = \lambda^d W(\{\varphi_k\}), \tag{36}$$

where the integers w_k, $k = 1, \ldots, \ell$, indicate the scaling property of the fields $\{\varphi_k\}$, while d denotes the degree of the superpotential. The vanishing condition $W = 0$ thus corresponds to a hypersurface in the weighted projective space $\mathbf{WCP}^{\ell-1}_{(w_1,\ldots,w_\ell)}$.

Motivated by this and with reference to (34), we thus introduce the new fields $\{y_i^a\}$, related to $\{Y_i^a\}$ by

$$\begin{aligned} y_i^1 &= \mathrm{e}^{\frac{Q_1^2-Q_2^2}{D}w_i^1 Y_i^1}, \qquad & i &= 1,2, \\ y_i^2 &= \mathrm{e}^{\frac{Q_2^1-Q_1^1}{D}w_i^2 Y_i^2}, \qquad & i &= 1,2,3. \end{aligned} \tag{37}$$

For these mappings to be sensible and do the intended job, we must assume that $Q_1^1 \neq Q_2^1$ and $Q_1^2 \neq Q_2^2$. These assumptions may therefore be interpreted as an initial requirement for (34) to correspond to a super-LG model. It is noted that they are neither necessary nor sufficient conditions for the non-vanishing of the determinant D. The path integral now reads

$$\begin{aligned} \mathcal{Z} \propto \int \left(\prod_{i=1}^{2} \mathrm{d}y_i^1\right)\left(\prod_{i=1}^{3} \mathrm{d}y_i^2\right) \exp\Bigg(\sum_{i=1}^{2}\left(y_i^1\right)^{\frac{D}{w_i^1(Q_2^2-Q_1^2)}} + \sum_{i=1}^{3}\left(y_i^2\right)^{\frac{D}{w_i^2(Q_1^1-Q_2^1)}} \\ + \mathrm{e}^{\frac{Q_2^2 t^1 - Q_2^1 t^2}{D}}\left(y_1^1 y_2^1\right)^{\frac{Q_2^2}{Q_2^2-Q_1^2}}\left(y_1^2 y_2^2 y_3^2\right)^{\frac{Q_2^1}{Q_2^1-Q_1^1}} + \mathrm{e}^{\frac{Q_1^1 t^2 - Q_1^2 t^1}{D}}\left(y_1^1 y_2^1\right)^{\frac{Q_1^2}{Q_1^2-Q_2^2}}\left(y_1^2 y_2^2 y_3^2\right)^{\frac{Q_1^1}{Q_1^1-Q_2^1}}\Bigg), \end{aligned} \tag{38}$$

and the vanishing of the superpotential turns into the algebraic equation

$$\begin{aligned} 0 = \sum_{i=1}^{2}\left(y_i^1\right)^{\frac{D}{w_i^1(Q_2^2-Q_1^2)}} + \sum_{i=1}^{3}\left(y_i^2\right)^{\frac{D}{w_i^2(Q_1^1-Q_2^1)}} + \mathrm{e}^{\frac{Q_2^2 t^1 - Q_2^1 t^2}{D}}\left(y_1^1 y_2^1\right)^{\frac{Q_2^2}{Q_2^2-Q_1^2}}\left(y_1^2 y_2^2 y_3^2\right)^{\frac{Q_2^1}{Q_2^1-Q_1^1}} \\ + \mathrm{e}^{\frac{Q_1^1 t^2 - Q_1^2 t^1}{D}}\left(y_1^1 y_2^1\right)^{\frac{Q_1^2}{Q_1^2-Q_2^2}}\left(y_1^2 y_2^2 y_3^2\right)^{\frac{Q_1^1}{Q_1^1-Q_2^1}}. \end{aligned} \tag{39}$$

In order to determine the appropriately associated weighted projective space, we consider the exponents of y_i^1 and y_i^2 in the two sums. Let

$$g = \gcd\left(w_1^1(Q_2^2-Q_1^2), w_2^1(Q_2^2-Q_1^2), w_1^2(Q_1^1-Q_2^1), w_2^2(Q_1^1-Q_2^1), w_3^2(Q_1^1-Q_2^1)\right) \tag{40}$$

denote the greatest common divisor of the denominators of these five exponents. The weighted projective space is then given by

$$\mathbf{WCP}^4_{\left(\frac{w_1^1(Q_2^2-Q_1^2)}{g},\frac{w_2^1(Q_2^2-Q_1^2)}{g},\frac{w_1^2(Q_1^1-Q_2^1)}{g},\frac{w_2^2(Q_1^1-Q_2^1)}{g},\frac{w_3^2(Q_1^1-Q_2^1)}{g}\right)}\left(y_1^1, y_2^1, y_1^2, y_2^2, y_3^2\right). \tag{41}$$

The superpotential (39) is now quasi-homogeneous, provided the remaining Kähler-dependent terms also have degree D/g, where D is the determinant (33). This is the case *exactly provided* the super-CY conditions (27) are satisfied, which in the present example, reduce to

$$w_1^1 + w_2^1 = Q_1^1 + Q_2^1, \qquad w_1^2 + w_2^2 + w_3^2 = Q_1^2 + Q_2^2. \tag{42}$$

This illustrates the announced correspondence between the super-CY conditions of the A-model and quasi-homogeneity of the B-model, and that the degree of the associated superpotential is given in terms of the determinant of the fermion charge matrix of the A-model. This is a new relation between a pair of mirror supermanifolds.

It is noted that the LG superpotential given by (39) is in general not polynomial and may include non-integer powers of the coordinates y. Let us examine when it *does* correspond to a polynomial for *generic* t^1 and t^2. That is, the two terms multiplied by the exponential expressions in the Kähler parameters are present. Particular correlated limits of these parameters could eliminate these terms and the conditions for polynomial behaviour accompanying them. The strong correlation with the super-CY condition, which has been derived for generic Kähler parameters, may therefore be lost in certain limits.

For (39) to be polynomial, we must require that all powers be non-negative integers. It follows from a comparison of the powers of $(y_1^1 y_2^1)$ in the two Kähler-dependent terms that either Q_2^2 or Q_1^2 must vanish. Likewise, from the powers of $(y_1^2 y_2^2 y_3^2)$ we find that either Q_1^1 or Q_2^1 vanishes. Since $D \neq 0$, we then have the two possibilities: (I) $Q_1^2 = Q_2^1 = 0$ or (II) $Q_1^1 = Q_2^2 = 0$. From the Kähler-independent terms, it then follows that (I) $Q_1^1 / w_i^1 \in \mathbb{Z}_>$ and $Q_2^2 / w_i^2 \in \mathbb{Z}_>$ or (II) $Q_2^1 / w_i^1 \in \mathbb{Z}_>$ and $Q_1^2 / w_i^2 \in \mathbb{Z}_>$. By imposing the super-CY condition (or quasi-homogeneity) as well, it follows in both cases, (I) and (II), that $w_1^1 = w_2^1$ and $(w_1^2, w_2^2, w_3^2) \in \{P(k, k, k), P(k, k, 2k), P(k, 2k, 3k)\}$, where P denotes a permutation and k is a non-vanishing integer. As discussed above, the greatest common divisor is conventionally one, limiting the possible values to $k = \pm 1$. Note that in these considerations, the sign of the determinant is related to the signs of the weight vectors (w_1^1, w_2^1) and (w_1^2, w_2^2, w_3^2) in the A-model. A homogeneous and polynomial structure thus arises in the B-model when the A-model is based on $\mathbf{CP}^1 \times \mathbf{WCP}^2_{(-1,-3,-2)}$, for example, provided the fermionic extension is governed by $Q_1^1 = Q_2^2 = 0$, $Q_2^1 = 2$ and $Q_1^2 = -6$. The degree of the polynomial is then $D/2 = 6$, and it describes a hypersurface in the compact, weighted projective space $\mathbf{WCP}^4_{(3,3,1,3,2)}(y_1^1, y_2^1, y_1^2, y_2^2, y_3^2)$. A simpler example arises when choosing to base the A-model on $\mathbf{CP}^1 \times \mathbf{CP}^2$ with fermionic extension given by $Q_1^1 = 2$, $Q_2^2 = 3$ and $Q_1^2 = Q_2^1 = 0$. The bosonic structure of the B-model is then described by a polynomial hypersurface of degree 6 in $\mathbf{WCP}^4_{(3,3,2,2,2)}(y_1^1, y_2^1, y_1^2, y_2^2, y_3^2)$.

There are several possible generalizations of the above analysis. We shall discuss some of them below, while others will be addressed elsewhere.

3.2. *Mirrors of fermionic extensions of* $\mathbf{CP}^1 \times \mathbf{CP}^{p-1}$

A first and simple generalization is to consider a sigma model whose target space is a fermionic extension of $\mathbf{CP}^1 \times \mathbf{CP}^{p-1}$, $p \geqslant 2$. This corresponds to a $U(1) \otimes U(1)$ gauge theory with $p + 2$ bosonic fields Φ_i and (in the quadratic case where $n = f$) two fermionic fields Ψ_α with charges

$$q'^1 = \big(1, 1, 0, 0, \ldots, 0 \,\big|\, Q_1^1, Q_2^1\big), \qquad q'^2 = \big(0, 0, 1, 1, \ldots, 1 \,\big|\, Q_1^2, Q_2^2\big). \tag{43}$$

The D-term constraints of this A-model are given by

$$\begin{aligned} &|\Phi_1|^2 + |\Phi_2|^2 + Q_1^1|\Psi_1|^2 + Q_2^1|\Psi_2|^2 = \Re(t^1), \\ &\sum_{i=3}^{p+2} |\Phi_i|^2 + Q_1^2|\Psi_1|^2 + Q_2^2|\Psi_2|^2 = \Re(t^2), \end{aligned} \tag{44}$$

while the super-CY conditions read

$$Q_1^1 + Q_2^1 = 2, \qquad Q_1^2 + Q_2^2 = p. \tag{45}$$

Now, following the prescription above, we introduce the field redefinitions

$$\begin{aligned} y_i &= \mathrm{e}^{\frac{Q_1^2 - Q_2^2}{D} Y_i}, \qquad & i &= 1, 2, \\ y_i &= \mathrm{e}^{\frac{Q_2^1 - Q_1^1}{D} Y_i}, \qquad & i &= 3, \ldots, p + 2, \end{aligned} \tag{46}$$

resulting in a B-model superpotential whose vanishing condition is given by

$$\begin{aligned} 0 = {} & y_1^{\frac{D}{Q_2^2 - Q_1^2}} + y_2^{\frac{D}{Q_2^2 - Q_1^2}} + \sum_{i=3}^{p+2} y_i^{\frac{D}{Q_1^1 - Q_2^1}} + \mathrm{e}^{\frac{Q_2^2 t^1 - Q_2^1 t^2}{D}} (y_1 y_2)^{\frac{Q_2^2}{Q_2^2 - Q_1^2}} (y_3 \cdots y_{p+2})^{\frac{Q_2^1}{Q_2^1 - Q_1^1}} \\ & + \mathrm{e}^{\frac{Q_1^1 t^2 - Q_1^2 t^1}{D}} (y_1 y_2)^{\frac{Q_1^2}{Q_1^2 - Q_2^2}} (y_3 \cdots y_{p+2})^{\frac{Q_1^1}{Q_1^1 - Q_2^1}}. \end{aligned} \tag{47}$$

With

$$D = Q_1^1 Q_2^2 - Q_1^2 Q_2^1, \qquad g = \gcd\big(Q_2^2 - Q_1^2, Q_1^1 - Q_2^1\big) \tag{48}$$

and the super-CY conditions (45) imposed, we find that (47) corresponds to a hypersurface of degree D/g in the weighted projective space $\mathbf{WCP}^{p+1}_{\left(\frac{Q_2^2-Q_1^2}{g},\frac{Q_2^2-Q_1^2}{g},\frac{Q_1^1-Q_2^1}{g},\ldots,\frac{Q_1^1-Q_2^1}{g}\right)}(y_1,\ldots,y_{p+2})$. The conditions for the superpotential to be polynomial are (I) $Q_1^2 = Q_2^1 = 0$ or (II) $Q_1^1 = Q_2^2 = 0$. It is noted that with the super-CY conditions imposed, $g = 1$ for p odd, while $g = 2$ for p even.

Let us analyse the two options for a polynomial superpotential, namely (I) or (II). It turns out that in either case, the polynomial and quasi-homogeneous superpotential reads

$$0 = y_1^2 + y_2^2 + \sum_{i=3}^{p+2} y_i^p + \mathrm{e}^{t^1/2} y_1 y_2 + \mathrm{e}^{t^2/p} y_3 \cdots y_{p+2} \tag{49}$$

and describes a hypersurface of degree $2p/g$ in

$$\begin{aligned} &\mathbf{WCP}^{p+1}_{(p,p,2,\ldots,2)}(y_1,\ldots,y_{p+2}), \qquad p \text{ odd},\\ &\mathbf{WCP}^{p+1}_{(\frac{p}{2},\frac{p}{2},1,\ldots,1)}(y_1,\ldots,y_{p+2}), \qquad p \text{ even}. \end{aligned} \tag{50}$$

That is, the degree is $2p$ for p odd and p for p even. This infinite family of weighted projective spaces has already appeared in the literature [12]. There[4] it is discussed that a quasi-homogeneous hypersurface of degree $2p$ (for p odd) or p (for p even) in the space (50) should be of relevance to mirror symmetry of higher dimensional manifolds. This is confirmed here since we have found that such hypersurfaces correspond to bosonic structures of supermanifolds which are mirror partners to fermionic extensions of $\mathbf{CP}^1 \times \mathbf{CP}^{p-1}$.

4. Mirrors of fermionic extensions of toric varieties

Now we extend our study of super-projective spaces to fermionic extensions of general toric varieties. With reference to the description at the beginning of section 3, in particular, the A-model is based on the $p+n$ bosonic fields Φ_i and the f fermionic fields Ψ_α with $U(1)^{\otimes n}$ charges

$$q'^a = (q^a \mid Q^a) = \big(q_1^a,\ldots,q_{p+n}^a \,\big|\, Q_1^a,\ldots,Q_f^a\big), \qquad a = 1,\ldots,n. \tag{51}$$

The extended D-term constraint equations of the present A-model read

$$\sum_{i=1}^{p+n} q_i^a |\Phi_i|^2 + \sum_{\alpha=1}^{f} Q_\alpha^a |\Psi_\alpha|^2 = \Re(t^a), \qquad a = 1,\ldots,n. \tag{52}$$

The associated mirror B-model is obtained in the same way as above, and the super-LG path integral becomes

$$\begin{aligned} \mathcal{Z} = \int \left(\prod_{i=1}^{p+n} \mathrm{d}Y_i\right)\left(\prod_{\alpha=1}^{f} \mathrm{d}X_\alpha\, \mathrm{d}\eta_\alpha\, \mathrm{d}\chi_\alpha\right) \delta\left(\sum_{i=1}^{p+n} q_i^1 Y_i - \sum_{\alpha=1}^{f} Q_\alpha^1 X_\alpha - t^1\right)\\ \times \cdots \times \delta\left(\sum_{i=1}^{p+n} q_i^n Y_i - \sum_{\alpha=1}^{f} Q_\alpha^n X_\alpha - t^n\right)\\ \times \exp\left(\sum_{i=1}^{p+n} \mathrm{e}^{-Y_i} + \sum_{\alpha=1}^{f} \mathrm{e}^{-X_\alpha}(1+\eta_\alpha\chi_\alpha)\right). \end{aligned} \tag{53}$$

[4] Here the family is labelled by p, $p \geqslant 2$, which in [12] is denoted by $n+1$.

Following the same procedure as before, we integrate out the $2f$ fermionic fields yielding

$$\int \left(\prod_{\alpha=1}^{f} \mathrm{d}\eta_\alpha \,\mathrm{d}\chi_\alpha\right) \exp\left(\sum_{\alpha=1}^{f} \mathrm{e}^{-X_\alpha}(1+\eta_\alpha\chi_\alpha)\right) = \left(\prod_{\alpha=1}^{f} \mathrm{e}^{-X_\alpha}\right) \exp\left(\sum_{\alpha=1}^{f} \mathrm{e}^{-X_\alpha}\right). \tag{54}$$

Focusing on the interesting situation where $n = f$ and where the delta functions appearing in (53) are eliminated by integrating out the fields X_α, the set of linear equations expressing the X fields in terms of the Y fields is given by

$$\sum_{\alpha=1}^{f} Q_\alpha^a X_\alpha = \sum_{i=1}^{p+f} q_i^a Y_i - t^a, \qquad a = 1, \ldots, f. \tag{55}$$

There is a unique solution for this system of equations, provided the determinant of the quadratic $f \times f$ matrix of fermion charges Q,

$$D = \det(Q), \tag{56}$$

is non-vanishing. We shall assume this. The solution to (55) is then given in terms of the inverse matrix Q^{-1} as it may be written

$$X_\alpha = \sum_{a=1}^{f} (Q^{-1})_\alpha^a \left(\sum_{i=1}^{p+f} q_i^a Y_i - t^a\right). \tag{57}$$

Note that if a is interpreted as the row index in Q_α^a, as in (25), it corresponds to the column index in $(Q^{-1})_\alpha^a$. After integrating out these fields, the path integral (53) is free of delta functions:

$$\begin{aligned} \mathcal{Z} \propto \int \left(\prod_{i=1}^{p+f} \mathrm{d}Y_i\right) &\exp\left(-\sum_{\alpha=1}^{f}\sum_{a=1}^{f}\sum_{i=1}^{p+f} (Q^{-1})_\alpha^a q_i^a Y_i\right) \\ &\times \exp\left(\sum_{i=1}^{p+f} \mathrm{e}^{-Y_i} + \sum_{\alpha=1}^{f} \exp\left(-\sum_{a=1}^{f} (Q^{-1})_\alpha^a \left(\sum_{i=1}^{p+f} q_i^a Y_i - t^a\right)\right)\right). \end{aligned} \tag{58}$$

In order to extract information on the underlying geometry, we again follow the prescription outlined in section 3. We therefore introduce the field redefinitions

$$y_i = \exp\left(-\sum_{\alpha=1}^{f}\sum_{a=1}^{f} (Q^{-1})_\alpha^a q_i^a Y_i\right), \tag{59}$$

and require that

$$\sum_{\alpha=1}^{f}\sum_{a=1}^{f} (Q^{-1})_\alpha^a q_i^a \neq 0, \qquad i = 1, \ldots, p+f. \tag{60}$$

This ensures, in particular, that the superpotential may be written as a finite sum of products of powers of the fields. The path integral (58) now reads

$$\begin{aligned} \mathcal{Z} \propto \int \left(\prod_{i=1}^{p+f} \mathrm{d}y_i\right) \exp\Bigg(&\sum_{i=1}^{p+f} y_i^{1/\{\sum_{\alpha=1}^{f}\sum_{a=1}^{f}(Q^{-1})_\alpha^a q_i^a\}} \\ &+ \sum_{\alpha=1}^{f} \mathrm{e}^{\sum_{c=1}^{f}(Q^{-1})_\alpha^c t^c} \prod_{i=1}^{p+f} y_i^{\{\sum_{a=1}^{f}(Q^{-1})_\alpha^a q_i^a\}/\{\sum_{\beta=1}^{f}\sum_{b=1}^{f}(Q^{-1})_\beta^b q_i^b\}}\Bigg). \end{aligned} \tag{61}$$

The vanishing of the superpotential thus defined may be characterized by a hypersurface in the weighted projective space

$$\mathbf{WCP}^{p+n-1}_{\left(\frac{D\sum_{\alpha=1}^{f}\sum_{a=1}^{f}(Q^{-1})^{a}_{\alpha}q^{a}_{1}}{g},\ldots,\frac{D\sum_{\alpha=1}^{f}\sum_{a=1}^{f}(Q^{-1})^{a}_{\alpha}q^{a}_{p+n}}{g}\right)}(y_1,\ldots,y_{p+n}), \tag{62}$$

where we have introduced the parameter

$$g=\gcd\left(D\sum_{\alpha=1}^{f}\sum_{a=1}^{f}(Q^{-1})^{a}_{\alpha}q^{a}_{1},\ldots,D\sum_{\alpha=1}^{f}\sum_{a=1}^{f}(Q^{-1})^{a}_{\alpha}q^{a}_{p+n}\right). \tag{63}$$

The hypersurface is given by the algebraic equation

$$0=\sum_{i=1}^{p+f}y_i^{1/\{\sum_{\alpha=1}^{f}\sum_{a=1}^{f}(Q^{-1})^{a}_{\alpha}q^{a}_{i}\}}+\sum_{\alpha=1}^{f}\mathrm{e}^{\sum_{c=1}^{f}(Q^{-1})^{c}_{\alpha}t^{c}}\prod_{i=1}^{p+f}y_i^{\{\sum_{a=1}^{f}(Q^{-1})^{a}_{\alpha}q^{a}_{i}\}/\{\sum_{\beta=1}^{f}\sum_{b=1}^{f}(Q^{-1})^{b}_{\beta}q^{b}_{i}\}}. \tag{64}$$

Note that the factors of the determinant D (56) in the definition of the weights in (62) are required, in general, to ensure that the weights are integers. The expression (64) is seen to be quasi-homogeneous provided

$$\sum_{i=1}^{p+f}q^{a}_{i}=\sum_{\alpha=1}^{f}Q^{a}_{\alpha},\qquad a=1,\ldots,f, \tag{65}$$

which are the super-CY conditions of the original fermionic extension of the projective variety in the A-model. The degree of the superpotential is then given by D/g. This provides the most general version presented here of the new correspondence between two supermanifolds paired by mirror symmetry.

The question of when the superpotential is polynomial is more complicated in this general case than in the projective example in section 3. It is beyond the scope of the present work to attempt such a classification, though we intend to address it elsewhere.

Instead, let us point out that the family of complex three-dimensional toric varieties $\tilde{F}_m$, $m\geqslant 0$, is covered by our analysis. That is, one may start with an A-model constructed as a fermionic extension of the toric variety $\tilde{F}_m$. It is characterized by the charge vectors

$$q'^{1}=\left(1,1,0,0,m\,\middle|\,Q^{1}_{1},Q^{1}_{2}\right),\qquad q'^{2}=\left(0,0,1,1,1\,\middle|\,Q^{2}_{1},Q^{2}_{2}\right), \tag{66}$$

with respect to the gauge group $U(1)^{\otimes 2}$. Imposing the super-CY conditions yields

$$Q^{1}_{1}+Q^{1}_{2}=m+2,\qquad Q^{2}_{1}+Q^{2}_{2}=3. \tag{67}$$

The superpotential (64) reduces to

$$\begin{aligned}0={}&y_1^{\frac{D}{Q^2_2-Q^2_1}}+y_2^{\frac{D}{Q^2_2-Q^2_1}}+y_3^{\frac{D}{Q^1_1-Q^1_2}}+y_4^{\frac{D}{Q^1_1-Q^1_2}}+y_5^{\frac{D}{Q^1_1-Q^1_2+m(Q^2_2-Q^2_1)}}\\&+\mathrm{e}^{\frac{Q^2_2t^1-Q^1_2t^2}{D}}(y_1y_2)^{\frac{Q^2_2}{Q^2_2-Q^2_1}}(y_3y_4)^{\frac{Q^1_2}{Q^1_2-Q^1_1}}y_5^{\frac{Q^1_2-mQ^2_2}{Q^1_2-Q^1_1-m(Q^2_2-Q^2_1)}}\\&+\mathrm{e}^{\frac{Q^1_1t^2-Q^2_1t^1}{D}}(y_1y_2)^{\frac{Q^2_1}{Q^2_1-Q^2_2}}(y_3y_4)^{\frac{Q^1_1}{Q^1_1-Q^1_2}}y_5^{\frac{Q^1_1-mQ^2_1}{Q^1_1-Q^1_2+m(Q^2_2-Q^2_1)}}\end{aligned} \tag{68}$$

and corresponds to a hypersurface in (62) which now reads

$$\mathbf{WCP}^{4}_{\left(\frac{Q^2_2-Q^2_1}{g},\frac{Q^2_2-Q^2_1}{g},\frac{Q^1_1-Q^1_2}{g},\frac{Q^1_1-Q^1_2}{g},\frac{Q^1_1-Q^1_2+m(Q^2_2-Q^2_1)}{g}\right)}(y_1,y_2,y_3,y_4,y_5), \tag{69}$$

where

$$g=\gcd\left(Q^2_2-Q^2_1,Q^1_1-Q^1_2,Q^1_1-Q^1_2+m\left(Q^2_2-Q^2_1\right)\right)=\gcd\left(Q^2_2-Q^2_1,Q^1_1-Q^1_2\right). \tag{70}$$

The degree of the superpotential is D/g. A simple adaptation of the discussion of the polynomial behaviour of the superpotential (39) reveals that in order for (68) to be a homogeneous polynomial, we again have the two cases (I) and (II). In case (I), for example, where $Q_1^2 = Q_2^1 = 0$, it follows from a comparison of the powers of y_5 in the two Kähler-dependent terms that one of the three entities Q_1^1, Q_2^2 or m must vanish. Since $D \neq 0$, we see that $m = 0$. A similar argument applies to case (II). We may thus conclude that the only $\tilde{F}_m$ which can result in a homogeneous polynomial (68) is $\tilde{F}_0$.

5. Conclusion

We have discussed mirror symmetry of supermanifolds constructed as fermionic extensions of toric varieties. This has been achieved by studying fermionic extensions of linear sigma A-models and their T-dual super-LG B-models. The present work primarily concerns the quadratic case where $n = f$ (i.e., equal numbers of $U(1)$ factors and fermionic fields in the A-model), and focus has been on the bosonic structure arising after integrating out the fields in the B-model obtained by dualizing the fermionic fields in the A-model. We have found that quasi-homogeneity of the resulting toric data of the B-model is related to the super-CY conditions of the A-model supermanifold. Furthermore, the degree of the associated B-model superpotential is given in terms of the determinant of the A-model fermion charge matrix. Several special cases have been used as illustrations of our general results.

Natural extensions of the present work include the non-quadratic case where $n \neq f$. It is also of interest to understand the different patches of the bosonic B-model structure that would result after integrating out different bosonic fields than the ones introduced by the dualization of the fermionic fields in the A-model. One should also try to extract geometric information on the full supermanifold in the B-model, and not just the bosonic structure of it obtained after integrating out the fermionic fields. We hope to discuss all of these interesting problems in the future.

Acknowledgments

AB thanks R Ahl Laamara and P Resco for discussions, while JR thanks C Cummins for discussions. LBD and AS thank the program Protars III, CNRT D12/25, Rabat.

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